Problem 17

Prove that if n is a positive integer, then $7^n - 1$ is divisible by 6.

Solution

Prove this by using the principle of mathematical induction. Start by showing that the base case is true. If $n = 1$, then

$$\frac{7^n - 1}{6} = \frac{7^1 - 1}{6} = \frac{6}{6} = 1.$$

Since the right side is an integer, $7^1 - 1$ is divisible by 6. Now assume the inductive hypothesis; that is, $7^k - 1$ is divisible by 6,

$$\frac{7^k - 1}{6} = m,$$

where m and k are positive integers. The aim is to show that $7^{k+1} - 1$ is also divisible by 6.

$$7^k - 1 = 6m$$

$$7^k = 6m + 1$$

$$7(7^k) = 7(6m + 1)$$

$$7^{k+1} = 42m + 7$$

$$7^{k+1} - 1 = 42m + 6$$

$$7^{k+1} - 1 = 6(7m + 1)$$

$$\frac{7^{k+1} - 1}{6} = 7m + 1$$

Since $7m + 1$ is a positive integer, $7^{k+1} - 1$ is divisible by 6. Therefore, by the principle of mathematical induction, $7^n - 1$ is divisible by 6 if n is a positive integer.